Absolute Values

The definition of an absolute value is “the distance from zero on the number line”.

In the number line above, both –3 and 3 are three units away from zero. The negative sign just tells us which direction to go: negative is left and positive is right. The way we write these two things mathematically is like so:

\[ |–3| = 3 \]  \( \Leftarrow \) This means that “the distance that negative 3 is from zero is 3 units”

\[ |3| = 3 \]  \( \Leftarrow \) This means that “the distance that positive 3 is from zero is 3 units”.

Therefore, any time we have a variable inside of an absolute value, there is always the potential of having TWO answers for \( x \), not just one.

**Example 1:** Solve for \( x \): \[ |x| = 4 \]

This means that the distance of “\( x \)” is four units away from 0. There are two numbers that qualify: Either \( x = 4 \) or \( x = –4 \). So, we write our answer like this: \( \{–4, 4\} \).

Another way to look at this is by graphing. The “v” shape to the right is the graph of \( |x| \). The horizontal line is the graph of 4.

When we set these equal to one another, we are asking where these two graphs cross each other.

There are two places where these graphs cross: at \( x = 4 \) and \( x = –4 \), which is the same answer we got earlier.

Also, notice that when we’re changing signs to take the negative answer into account, we don’t change the signs of the stuff on the inside of the absolute value!

**Example 2:** Solve for \( x \): \[ |2x + 1| = 7 \]

Again, we can examine where the graphs \( |2x + 1| \) and 7 cross. From the graph, it looks like \( x = 3 \) or \( x = –4 \). But, how do we handle this mathematically?

\[ |2x + 1| = 7 \] means that “\( 2x + 1 \)” is 7 units away from 0 in either the positive direction OR the negative direction. So:

\[ 2x + 1 = 7 \] \hspace{1cm} \textbf{or} \hspace{1cm} \[ 2x + 1 = –7 \]

What do you notice about the shapes of the graphs of absolute values? ______________________________

This is always true, so we can anticipate the shape of the graph even if we don’t know the exact placement on the x-y grid.
Example 3: Solve for x: \(|-3x - 2| = 5\)

Notice that it doesn’t matter if the numbers inside the absolute value are positive or negative. The graph is still going to be a V shape and that V is still sitting on the x-axis.

Absolute Value Inequalities

An inequality just asks “where is one graph above another” (for “>”) or “where is one graph below another” (for “<”).

Example 4: Solve the inequality: \(|x| < 4\).

Here, we want to know “where is the absolute value graph below the 4 line?” Notice that the x-values 4 and –4 are still important, like in Example 1, because that’s where one graph changes from being “above” to “below” with respect to the other. I call these the “Points of Interest”.

We find our Points of Interest by changing our inequality sign to an equals sign!

From the graph, we can see our answer is when x is between –4 and 4, or that \(-4 < x < 4\). Mathematically, we can find our Points of Interest and then examine which interval is going to work.

Points of Interest: Where \(|x| = 4\).
\[ x = 4 \quad \text{or} \quad x = -4 \]

Notice how our Points of Interest break our number line up into 3 pieces. Now try plugging a number into our original inequality that represents each part of our broken-up number line. These are called “Test Points”.

Test Points: Try \(x = -5\) \hspace{1cm} Try \(x = 0\) \hspace{1cm} Try \(x = 5\)

So, our number line graph of the solution is this:

We generally write our final answer in interval notation like so: \((-4, 4)\).

Example 5: Solve the inequality: \(|x| \geq 3\).

Here, we want to know “where is the absolute value graph above the 3 line?” From our graph, we can see that this actually happens in two separate places: when \(x \leq -3\) or when \(x \geq 3\).

Points of Interest: Where \(|x| = 3\).
\[ x = 3 \quad \text{or} \quad x = -3 \]
Test Points: Now we try some points in our broken-up number line to see what happens:

Try $x = -4$  \hspace{1cm} Try $x = 0$  \hspace{1cm} Try $x = 4$

Graph of solution: \hspace{1cm} Answer in interval notation: $(−∞,−3] \cup [3,∞)$.  

One thing to notice is this: The answer for the absolute value inequalities will either be

The stuff on the outside  \hspace{1cm} OR \hspace{1cm} The stuff in the middle

assuming that the number on the other side of the equals sign is positive. (If it’s 0, you’ll get only one point; if it’s negative, you’ll get no solution.)

This allows us to only have to try one point in one interval in order to figure out which part needs to be shaded on our number line.

The endpoints are still determined by whether we’re dealing with $\geq$ and $\leq$ or $<$ and $>$. The symbols $\geq$ and $\leq$ have endpoints that are filled-in dots; the symbols $<$ and $>$ have endpoints that are circles.

Example 6: Solve the inequality: $|−2x − 4| ≥ 5$

Here, we want to know where the absolute value graph is above the 5 line.

Points of Interest: Where $|−2x − 4| = 5$  \hspace{1cm} (Notice the = sign!)

\[−2x − 4 = 5 \quad \text{or} \quad −2x − 4 = −5\]

So, instead of having to check an $x$-value in all three parts, we only have to check one of them. If the number works, so does the interval; if not, it’s the other(s) that work. I like using $x = 0$ because it makes things cancel out.

Test Point: Try $x = 0$.  \hspace{1cm} Graph of solution: \hspace{1cm} Interval Notation Answer:

Remember:

If we’re given $<$ or $>$ in the inequality, we use a circle $\bigcirc$ for the endpoints and parentheses (“(“ and “)”) for interval notation.

If we’re given $\leq$ or $\geq$, we use a filled-in dot $\bullet$ for the endpoints and square brackets (“[“ and “]”) for interval notation.
Example 7: Solve the inequality: $|4x + 1| < 21$

Here, we want to know where the absolute value graph is _________ the 21 line.

Points of Interest: Where ________________.

Test Point: Try $x = ______$  

Graph of solution:

Interval Notation Answer: __________________________

Example 8: Solve the inequality: $|5x – 3| > 8$

We want to know where the absolute value graph is _________ the 8 line.

Points of Interest: Where ________________.

Test Point: Try $x = ______$  

Graph of solution:

Interval Notation Answer: __________________________

Here are the steps, for those of you who are “step-by-step” people:
1) Find the Points of Interest by rewriting the inequality as an equation.
2) Plot the Points of Interest on a number line.
3) Pick an $x$-value for a Test Point and plug it into the original inequality to determine interval(s) that “work” for the inequality. The Test Point cannot be one of the Points of Interest.
4) On the number line, shade the correct interval(s) and plot the correct endpoints.
5) Write the answer in interval notation (based on the number line).