Solving Systems of Equations – Graphing

Any time we have two or more equations, we have what is called a system of equations. In our case, we are going to be interested in two linear equations. The goal of a system of equations is finding the place where the (x, y) value is the same for both lines, or where the lines cross. One way we can do this is by graphing.

Example 1: \[
\begin{align*}
\frac{y}{2} & = \frac{x}{2} + 3 \\
\frac{y}{2} & = \frac{x}{2} + 2
\end{align*}
\]

Example 2: \[
\begin{align*}
\frac{2x}{2} & = \frac{y}{2} + 5 \\
\frac{2x}{2} & = \frac{y}{2} + 7
\end{align*}
\]

Example 3: \[
\begin{align*}
\frac{2x}{4} & = \frac{y}{4} + 8 \\
\frac{4x}{4} & = \frac{y}{4} + 16
\end{align*}
\]

Notice that in example 1, we get two lines that cross at a single point. In example 2, we get two lines that are parallel, and in example 3, we get a single line because the lines cross everywhere (both lines are the same). These are the three possibilities with systems of equations.

You should notice something special about the slopes in examples 2 and 3. Graphing is a great way to get an idea of where your lines are going to cross, if at all, but sometimes it can only give us an estimate and not the exact answer. This happens when the equations cross at points that are not “nice”. Since the birth of computers, however, graphing has become easy enough that it’s worth doing the graphing using a graphing program, like MathGV, or a graphing calculator just to check to see that we’re on the right track with our answers.

The following pairs of lines are special. Describe what is special about each pair.

1. \[
\begin{align*}
3x - 5y & = 7 \\
3x - 5y & = 12
\end{align*}
\]

2. \[
\begin{align*}
x + 4y & = 11 \\
2x + 8y & = 22
\end{align*}
\]

In 1 and 2 above, the slopes are what make these lines special. Find the slope for each of the lines in the systems.

For #1: \(3x - 5y = 7\) \(m = \) \(3x - 5y = 12\) \(m = \)

For #2: \(x + 4y = 11\) \(m = \) \(2x + 8y = 22\) \(m = \)

What do you notice about the slopes in each system in 1 and 2?___________________

______________________________________________________________________
Solving Systems of Equations – Substitution

A method called substitution can be used instead of graphing in order to get a more accurate result. This works on the premise that if we know a value for either “x” or “y”, then we can plug it into the second equation. Substitution is best used when one of the coefficients of “x” or “y” in either equation is “1”.

Example 4: Solve the system:

\[
\begin{align*}
x &= 2 \\
x + y &= 9
\end{align*}
\]

Since we know that \(x = 2\), then we can substitute “2” in for “x” in the second equation like so:

\(2 + y = 9\) \hspace{1cm} \text{(Now, solve for “y” \ldots \text{ (y = 7)})}

Once you get that “y” is 7, and you know that “x” is 2, put the numbers in ordered pair form: (2, 7). This is where the two lines cross!

Now let’s check out example 1 from the graphing examples above using substitution.

Example 1:

\[
\begin{align*}
y &= 2x + 3 \\
y &= x + 2
\end{align*}
\]

Notice that we have solved for “y” in both equations. While doing this in both equations is not necessary, we can use it to our advantage.

\(2x + 3 = x + 2\) \hspace{1cm} Since “y” is equal to both \(2x + 3\) and \(x + 2\), then they are equal to each other! Then, we can solve for “x” and plug our value back into either equation to get “y”. It doesn’t matter which, as we should get the same “y” value either way.

Then, since \(x = \underline{\hspace{1cm}}\), we plug that value of “x” into one of our equations. \(y = x + 2\) is the easiest, so that’s what we’ll use: \(y = (\underline{\hspace{1cm}}) + 2 = \underline{\hspace{1cm}}\).

So, the point of intersection is (\underline{\hspace{1cm}}, \underline{\hspace{1cm}}).

Let’s try this same idea on Example 2. We can solve for “y” in either equation above since the coefficient of “y” is “1” in both of them, so I’ll use the first: \(2x + y = 5\)

Example 2:

\[
\begin{align*}
2x + y &= 5 \\
2x + y &= 7
\end{align*}
\]

\[\begin{align*}
2x + y &= 5 \\
-2x + \underline{\hspace{1cm}} &= -2x
\end{align*}\]

\(y = \underline{\hspace{1cm}} \hspace{1cm} \leftarrow \text{Now that we’ve found what “y” is equal to, let’s plug it for “y” in the second equation, } 2x + y = 7.\)

\(2x + (\underline{\hspace{1cm}}) = 7 \hspace{1cm} \text{Again, distribute as necessary and solve for “x”, if possible.} \)

In this case, however, see how the “x”’s cancel out? Here we ended up with a statement that is false. Therefore, there are no places where the lines cross. They are parallel.

Sometimes, we get a true statement instead of a false statement. In this case, the lines cross everywhere, or are the same line. This would be the case with Example 3.
If you notice, these are the exact same three possibilities that we had when we were looking at solving linear equations:

1) We get a value for “x”, like \( x = \ldots \). (And now also a value for “y”.)
2) We end up with a statement that is always true. (An Identity)
3) We end up with a statement that is never true (False – A Contradiction).

Using substitution on Example 3 is not fun. Here is why.

**Example 3:**

\[
\begin{align*}
2x - 3y &= 8 \\
4x - 6y &= 16
\end{align*}
\]

If we solve either of the equations in Example 3 for either “x” or “y”, then we end up with fractions. Blar! 😞

\(2x - 3y = 8\) is either \( y = \frac{2}{3} x - \frac{8}{3}\) or \( x = \frac{3}{2} y + 4\) (and so is \(4x - 6y = 16\), actually, since they are the same equation when simplified). Either way, substitution with fractions is not necessarily a good time. Fortunately, there is yet another way to do these that works every time and mostly eliminates having to work with fractions.

**Practice – Using Substitution on a System of Linear Equations**

Find where the following lines cross (solve the system) using substitution. Note: Some answers may be fractions.

1) \( \begin{align*}
y &= 9 \\
2x + y &= 7
\end{align*} \) (___, ___)

2) \( \begin{align*}
x &= 12 \\
-5x + 2y &= -4
\end{align*} \) (___, ___)

3) \( \begin{align*}
4x &= 4 \\
-9x - 2y &= 16
\end{align*} \) (___, ___)

4) \( \begin{align*}
-3x &= 15 \\
6x - y &= 17
\end{align*} \) (___, ___)

In numbers 3 – 4 above, you had to solve for “x” or “y” first by dividing both sides by the coefficient of the variable. In the problems below, you’re going to have to put your skills to work by solving for either “x” or “y” in one question and plugging that whole thing into the other. **It’s easiest to solve for a variable with a coefficient of “1”**! For example:

Solve the system: \( \begin{align*}
x + 2y &= 7 \\
3x + y &= 11
\end{align*} \) ➞ Solve for “x” in the top equation: \( x = 7 - 2y \)

(Coefficient of x is 1)
Now, we plug $7 - 2y$ in for “$x$” in the second equation (because that’s what “$x$” equals).

\[ x = 7 - 2y \]

\[
\begin{align*}
3x + y &= 11 \\
3(7 - 2y) + y &= 11
\end{align*}
\]

Now, we solve for $y$!

To get “$x$”, plug “$y$” into one of the equations and solve for “$x$”. We might as well use $x = 7 - 2y$

\[
\begin{align*}
3(7 - 2y) + y &= 11 \\
21 - 6y + y &= 11 \\
21 - 5y &= 11 \\
-5y &= -10 \\
y &= 2
\end{align*}
\]

\[
\begin{align*}
x &= 7 - 2y \\
x &= 7 - 2(2) \\
x &= 7 - 4 \\
x &= 3
\end{align*}
\]

So, the point of intersection is (3, 2)!

Find where the following lines cross (solve the system) using substitution:

5) \[
\begin{align*}
x - y &= 2 \\
2x + 3y &= -6
\end{align*}
\]

(____, ____)

6) \[
\begin{align*}
x + 2y &= 2 \\
7x + 10y &= 2
\end{align*}
\]

(____, ____)

Answers: 1) (-1, 9) 2) (12, 28) 3) (1, $-\frac{25}{2}$) 4) (-5, -47) 5) (0, -2) 6) (-4, 3)

**Solving Systems of Equations – Elimination**

Elimination involves eliminating either the “$x$” or “$y$” variable out of both equations so the other can be solved for. Before we get into that, let’s examine a few true number sentences. We’re going to add these equations up and down to see if we get another true statement.

\[
\begin{align*}
2 + 9 &= 11 \\
3 + 5 &= 8
\end{align*}
\]

Try it with some negatives! \[
\begin{align*}
-9 + 2 &= -7 \\
-2 - 12 &= -14
\end{align*}
\]

Adding up and down gives us another true statement! So, this is the idea we are going to use for variables.
Example 5: \[ \begin{align*}
2x + y &= 5 \\
x - y &= -2
\end{align*} \]
So, add these up and down. Notice how the “y” value cancels out completely? This also gives us an equation that has only one variable in it and we can solve that equation for x.

We get \( x = \ldots \).

To find y, we can use either equation. For comparative purposes, let’s use both.

Using \( 2x + y = 5 \):
\[
2(\ldots) + y = 5
\]
Using \( x - y = -2 \):
\[
(\ldots) - y = -2
\]

So, our point of intersection is \((\ldots, \ldots)\).

We can use this method for checking out Example 3 from our graphing examples.

Example 3: The thing here is that if we just add, then nothing will cancel out: we’ll get \( 6x - 9x = 24 \), which does nothing at all to help us.

So, before we go any further, we need to multiply either the top or the bottom equation by something that will make one of the variables cancel.

Let’s multiply the top equation by \(-2\). Doing so will make the “x”s cancel: \(-2(2x - 3y = 8) \rightarrow -4x + 6y = -16\).

\[
\begin{align*}
-4x + 6y &= -16 \\
4x - 6y &= 16
\end{align*}
\]
\[\leftarrow \text{So, here is what we now have. We can add these now! See how everything cancels out? That means these are the same line because “0 = 0” (a true statement) means the lines are the same.}\]

One advantage to the elimination method is that, unlike with substitution, it keeps us from having to use fractions in our computations. For example, if we use the elimination method and get a fraction for an answer of one of the variables, we can always use the elimination method again to find the value of the other variable. Another advantage is that it always gives accurate, exact answers with linear equations, unlike with graphing.

Example 6: Solve the system: \[ \begin{align*}
2x + 3y &= 6 \\
3x - 4y &= 7
\end{align*} \]

First, let’s eliminate “x”. This will give us a value for “y”.

Notice how this does not give us a nice value for y. Instead of plugging this value into one of the equations and trying to muddle through dealing with the fractions that result,
another option is to look back at the original system of equations and repeat the elimination, but eliminate “y” instead of “x”. This will give us a value for x.

The least common multiple of the coefficients of “y” (3 and 4) is 12. Since the 4 is already negative in the second equation, then we don’t have to worry about dealing with a negative sign this time.

So, the point of intersection of these two lines is (______, ______).

Sometimes, when students get an answer like this, they’re very uncomfortable with whether it’s right or not. Here’s where graphing comes in handy. Find the decimal equivalent of your fraction answer and compare it to the answer the graph shows.

\[ x \approx \underline{\hspace{1cm}}, \quad y \approx \underline{\hspace{1cm}} \]

Practice – Using Elimination on a System of Linear Equations
Solving the following systems of equations using elimination.

1) \[ \begin{align*}
    x + y &= 4 \\
    x - y &= 2
\end{align*} \]
2) \[ \begin{align*}
    x - 2y &= 3 \\
    -2x + 4y &= 1
\end{align*} \]
3) \[ \begin{align*}
    x - y &= 2 \\
    2x + 3y &= -6
\end{align*} \]

Answers: 1) (3, 1) 2) No solution 3) (0, –2)
Systems of Equations and Applications

**Example 1:** Jason and Cathy are ticket-sellers at their college play. Jason is selling student tickets for $3.00 each, and Cathy is selling non-student tickets for $6.50 each. If their total income for 37 tickets was $188.00, how many tickets did Jason sell? (Note: Don’t freak out about the decimals. That’s why we have calculators!)

**Example 2:** Bill fenced in a rectangular garden in his yard. The length of the garden is 6 feet longer than the width and the perimeter is 104 feet. What is the width of the garden?

**Example 3:** A 40% saline solution is to be mixed with a 60% saline solution in order to get 8 liters of a 55% saline solution. How much of the 60% solution should be used?