Slope and Lines

The slope of a line is a ratio that measures the incline of the line. As a result, the smaller the incline, the closer the slope is to zero and the steeper the incline, the farther the slope value is away from zero. Another thing to keep in mind is that the slope is the same value from point to point (or between any two points on the line). This is called “the average rate of change” because it describes how much the line changes from point to point.

There are three different ways to view slope, labeled “m”.

1) \( m = \frac{\text{rise}}{\text{run}} \) ← this is the graphical way

2) Given two points \((x_1, y_1)\) and \((x_2, y_2)\): \( m = \frac{y_2 - y_1}{x_2 - x_1} \)

3) Calculate how much \( y \) changes from point to point and compare that to how much \( x \) changes from point to point.

The third way requires a little explanation. Let’s say we’re given the points \((3, 6)\) and \((5, 9)\) and we want to find what the slope of the line is between them. I’m going to make a “t” chart. “\( \Delta x \)” means “change in \( x \)” and “\( \Delta y \)” means “change in \( y \)”, or how much “\( x \)” and “\( y \)” change from point to point.

<table>
<thead>
<tr>
<th>( \Delta x )</th>
<th>( x )</th>
<th>( y )</th>
<th>( \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+2</td>
<td>3</td>
<td>6</td>
<td>+3</td>
</tr>
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Then, my slope is \( \frac{\Delta y}{\Delta x} = \frac{3}{2} \). If we use the slope formula, we’ll get the same thing: \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 6}{5 - 3} = \frac{3}{2} \).

There are a couple of other things to notice about slopes. One is that if a slope is positive, it is going uphill from left to right; if a slope is negative, it is going downhill from left to right. Since the slope above is positive, it is going uphill from left to right. In other words, as “\( x \)” gets bigger, “\( y \)” also gets bigger. In a negative slope, as “\( x \)” gets bigger, “\( y \)” gets smaller. The line above has a positive slope, so it goes uphill from left to right. Let’s graph it!
Notice how, as we go from point to point on the graph, we can identify the rise to the run. If we go from (3, 6) to (5, 9), we get \( \Delta y = +3 \) and \( \Delta x = +2 \), so, again, \( \frac{\Delta y}{\Delta x} = \frac{3}{2} \). If we go from (5, 9) to (3, 6), though, see how we get the same thing? This time \( \Delta y = -3 \) and \( \Delta x = -2 \), so \( \frac{\Delta y}{\Delta x} = \frac{-3}{-2} = \frac{3}{2} \). This works because of the nature of ratios.

Let’s find the slope of the line between a few pairs of points.

1) (-2, -7) and (1, -1)
2) (3, 4) and (-3, 4)
3) (2, 5) and (2, 3)

We can also use the slope to find more points on the same line. Keep in mind that the slope is \( \frac{\Delta y}{\Delta x} = \frac{3}{2} \), so that if we increase “y” by 3 and increase “x” by 2, we get another point!

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
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</table>

One thing to notice about #2 above is that with a slope of “0”, our line is going to be horizontal. If we graph it, we’ll see that it’s “flat” and has no incline. In #3, we have an undefined slope, which gives us a vertical line. In this case, our line is so steep that we don’t have a number to represent its incline.
We can also examine the slope of a line based on the given equation of the line. For example, let \( f(x) = \frac{3}{4}x + 2 \). To find points on this line, we’re not going to use the standard -2, -1, 0, 1, 2. If we did that, we’d end up having fractions as point values, and that’s difficult to graph. It would be easier for us to pick \( x \)-values that will cancel with that 4 in the denominator. So, let’s pick multiples of 4.

\[
f(x) = y = \frac{3}{4}x + 2
\]

Also, check the slope out from point to point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
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<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
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</tbody>
</table>

What is the slope of this line? _________

Do you see that number represented in the equation \( y = \frac{3}{4}x + 2 \)?

Another thing to notice is that when \( x = 0 \), the \( y \)-value is 2. Do you see the number “2” represented in the equation \( y = \frac{3}{4}x + 2 \)? The form \( y = mx + b \) is called “slope-intercept form” because the coefficient of “\( x \)” is the slope and “\( b \)” is the \( y \)-intercept (the \( y \)-value when \( x = 0 \) – this is where the line crosses the \( y \)-axis). Notice how when we plug in “0” for \( x \), the “\( mx \)” portion of the equation disappears and we’re left with \( y = b \). So, “\( m \)” is our slope and “\( b \)” is our \( y \)-intercept.

Find the slope and a point on the following lines just by examination:

1) \( y = \frac{2}{5}x - 6 \)  
2) \( y = 4x + 3 \) 
3) \( y = -\frac{1}{8}x + 10 \) 
4) \( y = x - 7 \)

\( m = \) _____  \( m = \) _____  \( m = \) _____  \( m = \) _____ 
point _______  point _______  point _______  point _______

The neat thing is that once we get one point, we can use the slope to find other points on the line, either by using the \( \frac{\text{rise}}{\text{run}} \) on the graph or by using \( \Delta x \) and \( \Delta y \).

This also means that if we know the \( y \)-intercept and the slope, we can find the equation of the line easily.
Many times when we're dealing with applications, the y-intercept (or “b”) is the starting value. This is because the x-axis represents time in many applications, and when $x = 0$, it represents when time is 0, or the starting time. The y-value, then, is the amount of something we have at that starting time. Here's an example.

**Example:** A roof has a 0.5-inch layer of ice on it from a previous storm. Another ice storm begins to deposit ice at a rate of 0.25 inch per hour. Write a linear function that represents the thickness of the ice on the storm $x$ hours after the ice storm started.

The amount of ice that is already on the roof is our y-intercept (“b”) because it’s how much we started with, or the amount of ice after 0 hours. The average rate of change (or slope) is 0.25 inches. Therefore, our equation is $y = \underline{\text{_____________}}$

☆ As a side note, it’s important to notice that since a slope is a *rate* or *ratio*, then it can be identified in a word problem anytime we see something like “inches per hour” or “miles per gallon”. That word “per” is really the thing that identifies the slope for us.

Let’s practice writing equations if we know the slope and the y-intercept. Find the equation of the line having the given slope and passing through the given point.

1) $m = 3$, $(0, 4)$  2) $m = \frac{2}{9}$, $(0, -3)$  3) $m = -\frac{7}{4}$, $(0, 9)$  4) $m = 9$, $(0, -1)$

$y = \underline{\text{_____________}}$  $y = \underline{\text{_____________}}$  $y = \underline{\text{_____________}}$  $y = \underline{\text{_____________}}$

Another linear form that’s important is what is commonly called *standard form*. It looks like this: $Ax + By = C$. The way we turn standard form into slope-intercept form is simply by solving for $y$. Let’s practice that. Change the following into slope-intercept form.

1) $3x + 4y = 8$  2) $5x - 7y = 21$  3) $4x + 3y = 7$