Polynomials

A polynomial has variables such that the exponents attached to them are whole numbers. In other words, polynomials don’t have exponents that are negative, fractions, or roots.

Examples:

a) $5x^4$

b) $4x^6 - 5x^3 + 21$

c) $2.3xy^3 - 7.64x^4y^2 + 91x$

d) $x^{-3}\sqrt{3} + 5y^{18}$

Notice, the coefficients (numbers multiplied by the variables) of the variables can be as ugly as they want to be. It’s the exponents of the variables that must be 0, 1, 2, 3, 4, 5,...

Non-Examples:

a) $x^y\sqrt{x}$

b) $\frac{12}{x - 7}$

c) $\frac{x}{y^3}$

d) $x^{1.5}$

e) $x^{-4}$

Why are the above not polynomials?
a) b) c) d) e)

Adding and subtracting polynomials involves something mathematicians refer to as “combining like terms”. Like terms in polynomials have the same variables and the same exponents attached to those variables. The coefficient in front of those variables tells us how many of those variables we have added together.

(Multiplication is a shortcut for repeated addition!)

Examples:

a) $3x^4 + 7x^4$

b) $17xy^3 - 28xy^3$

c) $3x^3y - 5xy^3$
The Distributive Property: The distributive property is the result of the fact that multiplication is a shortcut for repeated addition.

Examine the following: \(2(x + 3) = (x + 3) + (x + 3) = 2x + 6\) when we combine like terms. Notice that a shorter way will work. We can multiply the “2” by both numbers inside the parentheses and will still get \(2x + 6\).

This idea works regardless of what is out front of (or, in fact, behind) the parentheses.

Examples: 
\(\text{a) } 3(x^2 + 6x - 5) \quad \text{b) } -2x^3(5x^2 + 6x^7 - 4x) \quad \text{c) } 4xy(7xy^2 - 2x^3y^6)\)

Multiplying polynomials – Many times, people associate multiplying polynomials with what is commonly referred to as the “FOIL” method. This is only partially true. Let’s first examine something we’ve been doing since about 4\(^{th}\) grade now: 12 \times 13. When we multiply these two numbers, notice how we multiply both the 1 and the 3 in the “13” by both the 1 and the 2 in the “12”:

\[
\begin{array}{c}
12 \\
\times 13 \\
\end{array}
\]

\[
\begin{array}{c}
6 \\
30 \\
20 \\
100 \\
\end{array} \quad \begin{array}{c}
\rightarrow 12 \quad \leftarrow \text{Usually this is not how we do this problem.} \\
\rightarrow 12 \quad \leftarrow \text{This way is fine, but the first way shows more clearly how everything on the bottom gets multiplied by everything on the top!} \\
\rightarrow 120 \quad \leftarrow \\
\rightarrow 156 \quad \leftarrow \\
\end{array}
\]

Extend this idea to polynomials. Try \((x + 2)(x + 3)\).
Some folks write it like to the right in order to help themselves remember to multiply everything on the bottom by everything on the top instead of using the FOIL method. See how it resembles the way we multiply 12 x 13, even the indenting on the second row? This is just an extended version of the distributive property.

\[ x + 2 \times x + 3 = 3x + 6 \]
\[ x^2 + 2x + 0 = x^2 + 5x + 6 \]

Another way to do this is called the “box method”, which doesn’t initially look easier, but can help with organization in larger problems. For \((x + 2)(x + 3)\), the box would look like this:

\[
\begin{array}{ccc}
  x & 2 \\
  x & \left[\begin{array}{c}
  x \\
  3 \\
\end{array}\right]
\end{array}
\]

Notice how we place our numbers in such a way that top is the entire first parentheses and the side is the entire second parentheses. You can put “+” signs to designate the values are positive, if you wish. When we have a negative value, we certainly put the negative there. Then we just multiply the sides of the four inner boxes and add up what we get for the answer. This works great for larger problems like \((2x + 3)(5x^2 - 7x + 9)\).

\[
\begin{array}{ccc}
  5x^2 & -7x & 9 \\
  2x & \left[\begin{array}{c}
  5x^2 \\
  -7x \\
  9 \\
\end{array}\right]
\end{array}
\]

Again, the 2x and 3 are multiplied by everything in the second parentheses (5x^2, -7x, and 9).