Composition of Functions

When we plug one function into another one, it’s a composition of functions. It essentially lets us perform two steps in one. Let’s look at an example involving shoe sizes in America compared with Europe. (This is an old example, so the actual size relationships may have changed!)

Let \( f(x) = \frac{1}{2}x - 14 \), where “\( x \)” is the Italian size of a shoe and “\( f(x) \)” is the British size. Also, let \( g(x) = 2x + 24 \), where “\( x \)” is the American shoe size and “\( g(x) \)” is the Italian shoe size.

So, what if I know that my American shoe size is 8 and I want to find out what my British shoe size is? I don’t have a function that compares American sizes to British sizes. (Yet!) But, what I can do is change my American size to Italian, and then go from Italian to British. The idea looks something like this:

American shoe size \( \rightarrow \) Italian shoe size \( \rightarrow \) British shoe size

\( g(x) \) is what lets me do the first thing (change from American to Italian) and \( f(x) \) lets me do the second (change from Italian to British). Okay, so I have American size 8. Plug into \( g(x) \).

\[
g(8) = 2(8) + 24 = _____ + 24 = ______ in Italian size
\]

Then, take the Italian size and plug it into \( f(x) \) in order to change it into British size:

\[
f(\_ ) = \frac{1}{2}(\_ ) - 14 = ______ - 14 = _______ in British size!
\]

So, an American size 8 is the same as a British size _____. Let’s actually try this with several other sizes to see if we can figure out a pattern that will give us the relationship between American and British sizes. Notice that we are plugging in the result \( g(x) \) into \( f(x) \) in the second chart.

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Can we find a relationship between the two equations? What is it?

This is a composition of functions. Another way we can do this is by plugging the entire function $g(x)$ into $f(x)$ instead of plugging in numbers one at a time. This is written $(f \circ g)(x)$ or $f(g(x))$.

\[
(f \circ g)(x) = f(g(x)) = f(2x + 24) = \frac{1}{2}( \quad ) - 14 = \quad - 14 = \quad
\]

This bypasses the whole two-step process! In this case, it doesn’t make sense to do $g(f(x))$ because that would be trying to go from Italian shoe size to British shoe size (with $f(x)$) and then from British to American, but $g(x)$ doesn’t give us from British to American! It gives us American to Italian.

We’re going to work a couple more examples so that we get used to the process in general.

Let’s check this out: $f(x) = 3x - 6$ and $g(x) = x + 5$

1) Find $(f \circ g)(x)$

2) $(g \circ f)(x)$

3) $(g \circ g)(x)$

4) $(f \circ f)(x)$

When you compose a function with itself, like in numbers 3 and 4 above, it’s called iteration. If we were to compose a function with itself repeatedly, we could examine what the habit of that function would be over time and examine the patterns that result. Ever heard of “Chaos Theory”? Iteration is a Chaos Theory thing: there’s an entire branch of mathematics devoted to nothing but this! Cool, huh? 😊